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Orbit Raising with Low-Thrust Tangential Acceleration in Presence of Earth Shadow

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The problem of low-thrust tangential thrusting along small-to-moderate eccentricity orbits in the presence of Earth shadow is analyzed. Given the orbital elements and the shadow geometry at the start of each revolution, the changes in the in-plane orbit elements after one revolution of intermittent thrusting are evaluated analytically for a given level of constant acceleration. These perturbation equations are valid for small-to-moderate eccentricities ($0 \le e \le 0.2$), except for the argument of perigee, which is valid for any eccentricity larger than 0.01 due to the well-known singularity at $e = 0$ associated with the use of the classical elements. When e is less than 0.01, a nonsingular set of equations is used instead so that the orbit is continuously updated with negligible computational effort. These analytic guidance equations valid for low-thrust accelerations on the order of $10^{-4}\,g$ and less are developed for implementation in efficient transfer simulation programs for systems design optimization and preliminary mission analysis work. Furthermore, for the problem of continuous constant low-thrust tangential acceleration, the analytic integration of the orbit equations is shown to be accurate for several tens of revolutions in low Earth orbit and about 10 revolutions in geosynchronous Earth orbit. The analytic integration is further extended to include the effect of the Earth oblateness on the expanding orbit. This analytic long-term orbit prediction capability will minimize the computational loads of an onboard computer for autonomous orbit transfer applications and allow, among other things, the consideration of long multiorbit data arcs for analytic orbit determination updates, thereby decreasing considerably the frequency of these updates.

Nomenclature

E	= eccentric anomaly
f_r, f_θ, f_h	= perturbation acceleration vector components along radial, perpendicular, and out-of-plane directions
f_t, f_n	= tangent and normal components of acceleration vector
h	= orbit angular momentum
J_2	= second zonal harmonic, 1.082×10^{-3}
n	= spacecraft mean motion, $(\mu/a^3)^{1/2}$, rad/s
p	= orbit parameter
R	= equatorial radius of Earth, 6378.14 km
r	= radial distance
s_α, c_α	= $\sin\alpha$, $\cos\alpha$, etc.
α	= mean longitude, $\omega + M$
θ	= angular position measured from ascending node
θ^*	= true anomaly
μ	= gravity constant of Earth, 398,601.3, km^3/s^2
χ	= angle between radial direction and velocity vector

Introduction

THE optimal low-thrust transfer between a low Earth orbit (LEO) and a higher orbit not lying in the same plane is such that the orbit raising and inclination change are carried out simultaneously during the transfer with an optimal and varying mix of in-plane and out-of-plane accelerations. Furthermore, when shadowing is present during the initial phase of the transfer and due to the subsequent intermittent thrusting, the transfer orbit builds up eccentricity with apogee along the shadow arc. Numerical simulations show that for the most severe shadow geometry the eccentricity can build up from zero to a maximum of about 0.2, which makes problematic the use of most of the analytic or semianalytic optimal transfer guidance algorithms such as the Edelbaum[1] and Wiesel–Alfano[2] algorithms, which are valid for near-circular orbits and no shadowing. These algorithms are simple to use and to calculate, as opposed to the exact optimal transfer guidance algorithm of the SECKSPOT software,[3] which requires extensive computational capability. A theory of orbit raising in the presence of Earth shadow was developed[4] with the additional constraint that the eccentricity be kept at zero during the transfer. This will tend to lengthen slightly the total transfer time with a proportional increase in the propellant expenditure. However, these constrained guidance laws are calculated analytically, and once the orbit is in full sunlight, the Edelbaum or Wiesel–Alfano steering laws can once again be used to carry out the remaining part of the transfer. Furthermore, and as a drawback, these eccentricity-constrained strategies,[4] besides requiring longer transfer times, also result in increased dwell times in the Van Allen radiation belts, which further degrade the solar panels due to an increase in the total accumulated fluence. Thus it is more appropriate to carry out the first leg of the transfer by applying the thrust along the velocity vector without any inclination change such that the transfer is initially coplanar and the dwell time in the radiation belts is minimized. This strategy will result in the eccentricity buildup previously mentioned, as long as shadowing is present. Although semianalytic methods were developed for the discontinuous thrust transfer problem,[5,6] efficient codes to support systems optimization and preliminary mission analyses that require a large number of iterations require fast analytic guidance algorithms that also model all other important effects such as solar power output degradation due to the Van Allen radiation belts. An effort to create such a computer program was undertaken using circular orbit assumption and constant yaw profile during the transfer.[7] The strategy of coplanar tangential thrusting in the initial phases of the transfer shown here can easily be implemented in the aforementioned code regardless of whether shadowing is present. Once the orbit is safely outside the radiation belts, other analytic guidance laws can be applied to slowly raise, circularize, and rotate the orbit to reach the destination geostationary condition. This paper concentrates on the initial coplanar phase of the transfer and develops analytic expressions for the changes in the in-plane orbit elements after each revolution of intermittent tangential thrusting with constant acceleration. The orbit is thus updated from revolution to revolution until perigee is safely out of the belts, after which the orbit can be circularized

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following the same techniques of Ref. 4 and the transfer completed with simultaneous orbit raising and plane change.

General Analysis

The first-order differential equations for the classical in-plane orbital elements, semimajor axis, eccentricity, and argument of perigee are given in terms of the acceleration vector components along the instantaneous radial, orthogonal, and out-of-plane directions[8] by

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2na^2}{\mu}\left[\frac{aes_{\theta*}}{(1-e^2)^{\frac{1}{2}}} f_r + \frac{a^2(1-e^2)^{\frac{1}{2}}}{r} f_\theta\right] \tag{1}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{na^2}{\mu}(1-e^2)^{\frac{1}{2}}[s_{\theta*} f_r + (c_{\theta*} + c_E) f_\theta] \tag{2}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{na^2}{\mu e}(1-e^2)^{\frac{1}{2}}\left[-c_{\theta*} f_r + s_{\theta*}\left(1+\frac{r}{p}\right) f_\theta\right] - c_i\dot{\Omega} \tag{3}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{a^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}}}{\mu^{\frac{1}{2}}}\frac{(1+ec_{\theta*})^{-1}s_\theta f_h}{s_i} \tag{4}$$

In the preceding equations, $\theta = \omega + \theta^*$, $r = a(1-e^2)/(1+ec_{\theta*})$, and $p = a(1-e^2) = h^2/\mu$. These equations, which are widely used in the literature, have been specialized and converted to a form that is more convenient to analyze drag and thrust perturbation effects. If we assume that the thrust acceleration is essentially along the velocity vector, it is preferable to transform the equations of motion in Eqs. (1–3) so that they are given in terms of the tangential and normal acceleration components f_t and f_n, respectively. The component f_t is along the orbit tangent or velocity vector direction, whereas f_n is perpendicular to the tangent in the orbit plane and pointing away from the origin. The transformation between f_r, f_θ, and f_t, f_n is given by[8]

$$f_r = c_\chi f_t + s_\chi f_n \tag{5}$$

$$f_\theta = s_\chi f_t - c_\chi f_n \tag{6}$$

These equations can also be written as

$$f_r = \frac{e\mu s_{\theta*}}{hV} f_t + \frac{h}{rV} f_n \tag{7}$$

$$f_\theta = \frac{h}{rV} f_t - \frac{e\mu s_{\theta*}}{hV} f_n \tag{8}$$

with the velocity V given by

$$V^2 = \frac{(\mu/a)\left(1+e^2+2ec_{\theta*}\right)}{1-e^2} \tag{9}$$

This expression for the velocity in terms of the orbital elements is easily obtained from the energy equation $V^2/2 - \mu/r = -\mu/(2a)$, in which r is replaced by $a(1-e^2)(1+ec_{\theta*})^{-1}$. If Eqs. (7) and (8) are used in Eq. (1) and in view of $h = \mu^{1/2}a^{1/2}(1-e^2)^{1/2}$, the following simple form for $\mathrm{d}a/\mathrm{d}t$ can be obtained:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2na^3}{\mu(1-e^2)^{\frac{1}{2}}}\left(1+e^2+2ec_{\theta*}\right)^{\frac{1}{2}} f_t \tag{10}$$

and is valid for any elliptic orbit. This can also be cast into the form involving V

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2a^2}{\mu} V f_t \tag{11}$$

The f_n component has effectively canceled out as it should because it cannot contribute to any changes in the orbital energy. In a similar manner, Eq. (2) can also be transformed to the form involving f_t and f_n by first observing that, from the orbit equation $r = a(1-ec_E)$, it follows that $c_E = (1-r/a)/e = (e+c_{\theta*})/(1+ec_{\theta*})$. After some manipulation, we get

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{a^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}}}{\mu^{\frac{1}{2}}}\left[\frac{2(e+c_{\theta*})}{\left(1+e^2+2ec_{\theta*}\right)^{\frac{1}{2}}} f_t + \frac{s_{\theta*}(1-e^2)}{(1+ec_{\theta*})\left(1+e^2+2ec_{\theta*}\right)^{\frac{1}{2}}} f_n\right] \tag{12}$$

Finally, Eq. (3) for $\mathrm{d}\omega/\mathrm{d}t$ is transformed to

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{na^2(1-e^2)^{\frac{1}{2}}}{\mu e}\left[-c_{\theta*} f_r + \frac{(2+ec_{\theta*})s_{\theta*}}{1+ec_{\theta*}} f_\theta\right] \tag{13}$$

where $f_h = 0$ has been used to eliminate the $\dot{\Omega}$ contribution. Incidentally, expressions (7) and (8) can also be written as

$$f_r = \frac{es_{\theta*}}{\left(1+e^2+2ec_{\theta*}\right)^{\frac{1}{2}}} f_t + \frac{1+ec_{\theta*}}{\left(1+e^2+2ec_{\theta*}\right)^{\frac{1}{2}}} f_n \tag{14}$$

$$f_\theta = \frac{1+ec_{\theta*}}{\left(1+e^2+2ec_{\theta*}\right)^{\frac{1}{2}}} f_t - \frac{es_{\theta*}}{\left(1+e^2+2ec_{\theta*}\right)^{\frac{1}{2}}} f_n \tag{15}$$

Using f_r and f_θ in Eq. (13) results, after some manipulations, in

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{na^2(1-e^2)^{\frac{1}{2}}}{\mu e}\left[\frac{2s_{\theta*}}{\left(1+e^2+2ec_{\theta*}\right)^{\frac{1}{2}}} f_t - \frac{2e+c_{\theta*}(1+e^2)}{(1+ec_{\theta*})\left(1+e^2+2ec_{\theta*}\right)^{\frac{1}{2}}} f_n\right] \tag{16}$$

Finally, the perturbation of the true anomaly θ^* can be obtained from

$$\frac{\mathrm{d}\theta^*}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}t} - \frac{\mathrm{d}\omega}{\mathrm{d}t} \tag{17}$$

where

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{h}{r^2} - \dot{\Omega}c_i \tag{18}$$

Replacing Eqs. (3) and (18) in Eq. (17) will yield

$$\frac{\mathrm{d}\theta^*}{\mathrm{d}t} = \frac{h}{r^2} - \frac{r}{he}[-c_{\theta*}(1+ec_{\theta*})f_r + s_{\theta*}(2+ec_{\theta*})f_\theta] \tag{19}$$

which reduces to the following form involving f_t and f_n:

$$\frac{\mathrm{d}\theta^*}{\mathrm{d}t} = \frac{\mu^{\frac{1}{2}}a^{-\frac{3}{2}}}{(1-e^2)^{\frac{3}{2}}}(1+ec_{\theta*})^2 - \frac{a^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}}}{e\mu^{\frac{1}{2}}(1+ec_{\theta*})} \times \left[\frac{2s_{\theta*}(1+ec_{\theta*})}{\left(1+e^2+2ec_{\theta*}\right)^{\frac{1}{2}}} f_t - \frac{\left(2e+e^2c_{\theta*}+c_{\theta*}\right)}{\left(1+e^2+2ec_{\theta*}\right)^{\frac{1}{2}}} f_n\right] \tag{20}$$

which is equivalent to $\mathrm{d}\theta^*/\mathrm{d}t = (h/r^2) - \dot{\omega}$ because $\dot{\Omega} = 0$ and because only in-plane accelerations are applied. Equation (20) can be replaced by a perturbation equation for the mean anomaly M because from

$$\frac{\mathrm{d}M}{\mathrm{d}t} = n + \frac{1-e^2}{nae}\left[\left(c_{\theta*} - \frac{2e}{1+ec_{\theta*}}\right)f_r - \left(\frac{2+ec_{\theta*}}{1+ec_{\theta*}}\right)s_{\theta*} f_\theta\right] \tag{21}$$

it follows that, after transformation,

$$\frac{\mathrm{d}M}{\mathrm{d}t} = n - \frac{(1-e^2)a^{\frac{1}{2}}}{e\mu^{\frac{1}{2}}\left(1+e^2+2ec_{\theta^*}\right)^{\frac{1}{2}}} \times \left[\frac{2\left(1+ec_{\theta^*}+e^2\right)}{1+ec_{\theta^*}}s_{\theta^*}f_t - \frac{1-e^2}{1+ec_{\theta^*}}c_{\theta^*}f_n\right] \tag{22}$$

For each time t during the numerical integration, $M = n(t-t_0) + M_0$, where M_0 is the mean anomaly at time t_0. Then from Kepler's equation $M = E - es_E$, the eccentric anomaly E is computed, from which the true anomaly is evaluated by using

$$\tan(\theta^*/2) = [(1+e)/(1-e)]^{\frac{1}{2}}\tan(E/2) \tag{23}$$

If we consider only tangential thrusting, then $f_n = 0$ and the relevant equations of motion reduce to the set

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2na^3}{\mu(1-e^2)^{\frac{1}{2}}}\left(1+e^2+2ec_{\theta^*}\right)^{\frac{1}{2}}f_t \tag{24}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{2a^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}}}{\mu^{\frac{1}{2}}}\frac{(e+c_{\theta^*})}{\left(1+e^2+2ec_{\theta^*}\right)^{\frac{1}{2}}}f_t \tag{25}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{2na^2(1-e^2)^{\frac{1}{2}}}{\mu e}\frac{s_{\theta^*}}{\left(1+e^2+2ec_{\theta^*}\right)^{\frac{1}{2}}}f_t \tag{26}$$

$$\frac{\mathrm{d}\theta^*}{\mathrm{d}t} = \frac{n}{(1-e^2)^{\frac{3}{2}}}(1+ec_{\theta^*})^2 - \frac{2a^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}}}{e\mu^{\frac{1}{2}}}\frac{s_{\theta^*}}{\left(1+e^2+2ec_{\theta^*}\right)^{\frac{1}{2}}}f_t \tag{27}$$

Both $\mathrm{d}\omega/\mathrm{d}t$ and $\mathrm{d}\theta^*/\mathrm{d}t$ have singularities at $e = 0$ because they are divided by e. These equations then break down for near-circular orbits or $e \leq 10^{-3}$ but are otherwise valid for any eccentricity. Given a constant tangential acceleration f_t, these equations can be integrated numerically to provide the changes experienced by the in-plane classical elements as a function of time. They are a set of fully coupled, nonlinear, first-order differential equations in the osculating elements a, e, ω, and θ^*. If we neglect the osculation of θ^* due to the acceleration f_t, then we can change the independent variable from t to θ^* because then, from Eq. (27),

$$\mathrm{d}t = \frac{(1-e^2)^{\frac{3}{2}}}{n(1+ec_{\theta^*})^2}\mathrm{d}\theta^* \tag{28}$$

Replacing in Eqs. (24–26), we get

$$\mathrm{d}a = \frac{2(1-e^2)\left(1+e^2+2ec_{\theta^*}\right)^{\frac{1}{2}}}{n^2(1+ec_{\theta^*})^2}f_t\,\mathrm{d}\theta^* \tag{29}$$

$$\mathrm{d}e = \frac{2(1-e^2)^2(e+c_{\theta^*})}{an^2(1+ec_{\theta^*})^2\left(1+e^2+2ec_{\theta^*}\right)^{\frac{1}{2}}}f_t\,\mathrm{d}\theta^* \tag{30}$$

$$\mathrm{d}\omega = \frac{2a^2(1-e^2)^2 s_{\theta^*}}{\mu e(1+ec_{\theta^*})^2\left(1+e^2+2ec_{\theta^*}\right)^{\frac{1}{2}}}f_t\,\mathrm{d}\theta^* \tag{31}$$

These equations can now be integrated analytically if we hold a, e, and f_t constant in the right-hand side of the preceding expressions. The mean motion n is also held constant because it is a function of a.

Analytic Integration with Intermittent Thrusting

The acceleration f_t being on the order of $10^{-4}\,g$ and less, it is enough to evaluate the first-order effect on a, e, and ω by holding a, e, and f_t constant during the integration over one revolution. Then we can update the elements and integrate analytically over the next revolution and so on until the required transfer is achieved. There exists no closed-form solution for da and de, and therefore we must expand in powers of the small quantity the eccentricity before carrying out the integration. Let us observe that

$$\left(1+e^2+2ec_{\theta^*}\right)^{\frac{1}{2}} = (1+e^2)^{\frac{1}{2}}(1+a_1c_{\theta^*})^{\frac{1}{2}} \tag{32}$$

with $a_1 = 2e/(1+e^2)$. Because a_1 is on the order of e, let us expand the following terms up to $\mathcal{O}(e^4)$:

$$(1+a_1c_{\theta^*})^{\frac{1}{2}} \simeq 1 + \tfrac{1}{2}a_1c_{\theta^*} - \tfrac{1}{8}a_1^2c_{\theta^*}^2 + \tfrac{1}{16}a_1^3c_{\theta^*}^3 - \tfrac{5}{128}a_1^4c_{\theta^*}^4 + \mathrm{HOT}$$

where HOT indicates higher-order terms in the binomial expansion. Similarly,

$$(1+ec_{\theta^*})^{-2} \simeq 1 - 2ec_{\theta^*} + 3e^2c_{\theta^*}^2 - 4e^3c_{\theta^*}^3 + 5e^4c_{\theta^*}^4 + \mathrm{HOT}$$

Let

$$K = \frac{2(1-e^2)}{n^2}f_t \tag{33}$$

Then

$$\begin{aligned}\Delta a &= K\int_0^{2\pi}\frac{\left(1+e^2+2ec_{\theta^*}\right)^{\frac{1}{2}}}{(1+ec_{\theta^*})^2}\mathrm{d}\theta^* \\ &= K\int_0^{2\pi}\left(1+b_1c_{\theta^*}+b_2c_{\theta^*}^2+b_3c_{\theta^*}^3+b_4c_{\theta^*}^4\right)\mathrm{d}\theta^* \\ &= 2\pi K\left(1+\frac{b_2}{2}+\frac{3}{8}b_4\right)\end{aligned} \tag{34}$$

where

$$b_1 = \tfrac{1}{2}a_1 - 2e \tag{35}$$

$$b_2 = 3e^2 - a_1e - \tfrac{1}{8}a_1^2 \tag{36}$$

$$b_3 = \tfrac{3}{2}a_1e^2 - 4e^3 + \tfrac{1}{4}a_1^2e + \tfrac{1}{16}a_1^3 \tag{37}$$

$$b_4 = 5e^4 - 2a_1e^3 - \tfrac{3}{8}a_1^2e^2 - \tfrac{1}{8}a_1^3e - \tfrac{5}{128}a_1^4 \tag{38}$$

We can expand b_2 and b_4 in powers of e to get $b_2 \simeq e^2/2 + 3e^4$, $b_4 \simeq -17/8\,e^4$ yielding the final form for Δa after one revolution of continuous thrust

$$\Delta a = 2\pi K\left[1 + (e^2/4) + \tfrac{45}{64}e^4\right] \tag{39}$$

This equation is accurate for small-to-moderate eccentricity up to 0.2 with an error of less than 1%. If we wish to integrate between 0 and θ_1^* and then between θ_2^* and 2π, with θ_1^* and θ_2^* standing for shadow entry and exit true anomalies, respectively, then from

$$\Delta a = K\int_0^{\theta_1^*}\frac{\left(1+e^2+2ec_{\theta^*}\right)^{\frac{1}{2}}}{(1+ec_{\theta^*})^2}\mathrm{d}\theta^* + K\int_{\theta_2^*}^{2\pi}\frac{\left(1+e^2+2ec_{\theta^*}\right)^{\frac{1}{2}}}{(1+ec_{\theta^*})^2}\mathrm{d}\theta^*$$

we get an expression for Δa valid for intermittent thrusting due to the presence of Earth shadow along the $\theta_2^* - \theta_1^*$ arc:

$$\Delta a = 2\pi K\left(1 + \frac{b_2}{2} + \frac{3}{8}b_4\right) + K\Bigg\{\left(\theta_1^* - \theta_2^*\right) + b_1\left(s_{\theta_1^*} - s_{\theta_2^*}\right) + b_2\left[\frac{\theta_1^* - \theta_2^*}{2} + \frac{1}{4}\left(s_{2\theta_1^*} - s_{2\theta_2^*}\right)\right] + \frac{b_3}{3}\left[s_{\theta_1^*}\left(c^2_{\theta_1^*} + 2\right) - s_{\theta_2^*}\left(c^2_{\theta_2^*} + 2\right)\right] + b_4\left[\frac{3}{8}\left(\theta_1^* - \theta_2^*\right) + \frac{1}{4}\left(s_{2\theta_1^*} - s_{2\theta_2^*}\right) + \frac{1}{32}\left(s_{4\theta_1^*} - s_{4\theta_2^*}\right)\right]\Bigg\} \tag{40}$$

For the integration of the $\mathrm{d}e/\mathrm{d}t$ equation, we need the following additional expansions:

$$(1 + a_1 c_{\theta^*})^{-\frac{1}{2}} \simeq 1 - \tfrac{1}{2}a_1 c_{\theta^*} + \tfrac{3}{8}a_1^2 c^2_{\theta^*} - \tfrac{5}{16}a_1^3 c^3_{\theta^*} + \tfrac{35}{128}a_1^4 c^4_{\theta^*}$$

$$(1 + a_1 c_{\theta^*})^{-\frac{1}{2}}(1 + e c_{\theta^*})^{-2} \simeq 1 + c_1 c_{\theta^*} + c_2 c^2_{\theta^*} + c_3 c^3_{\theta^*} + c_4 c^4_{\theta^*}$$

where

$$c_1 = -2e - \tfrac{1}{2}a_1 \tag{41}$$

$$c_2 = 3e^2 + a_1 e + \tfrac{3}{8}a_1^2 \tag{42}$$

$$c_3 = -4e^3 - \tfrac{3}{2}a_1 e^2 - \tfrac{3}{4}a_1^2 e - \tfrac{5}{16}a_1^3 \tag{43}$$

$$c_4 = 5e^4 + 2a_1 e^3 + \tfrac{9}{8}a_1^2 e^2 + \tfrac{5}{8}a_1^3 e \tag{44}$$

Letting

$$K' = \frac{2(1 - e^2)^2}{an^2(1 + e^2)^{\frac{1}{2}}} f_t \tag{45}$$

and integrating Eq. (30) between 0 and 2π yields

$$\Delta e = K'\int_0^{2\pi} (e + c_{\theta^*})\left(1 + c_1 c_{\theta^*} + c_2 c^2_{\theta^*} + c_3 c^3_{\theta^*} + c_4 c^4_{\theta^*}\right)\mathrm{d}\theta^*$$

which, after some manipulations, reduces to

$$\Delta e = 2\pi K'\left[e\left(1 + c_2/2 + \tfrac{3}{8}c_4\right) + \left(c_1/2 + \tfrac{3}{8}c_3\right)\right] = -\pi K' e\left(1 + \tfrac{15}{8}e^2\right) \tag{46}$$

If $e = 0$, then Δe will vanish at the end of each revolution with continuous tangential thrusting. If $e \neq 0$, then Δe will be always negative from Eq. (46), indicating that the orbit will be less eccentric after each revolution with tangential continuous thrusting. If, on the other hand, we consider intermittent thrusting from 0 to θ_1^* and from θ_2^* to 2π, then

$$\Delta e = K'\Bigg\{e\left(\theta_1^* - \theta_2^*\right) + d_1\left(s_{\theta_1^*} - s_{\theta_2^*}\right) + d_2\left[\frac{\theta_1^* - \theta_2^*}{2} + \frac{1}{4}\left(s_{\theta_1^*} - s_{\theta_2^*}\right)\right] + d_3\left[2\left(s_{\theta_1^*} - s_{\theta_2^*}\right) + s_{\theta_1^*}c^2_{\theta_1^*} - s_{\theta_2^*}c^2_{\theta_2^*}\right] + d_4\left[\frac{3}{8}\left(\theta_1^* - \theta_2^*\right) + \frac{1}{4}\left(s_{2\theta_1^*} - s_{2\theta_2^*}\right) + \frac{1}{32}\left(s_{4\theta_1^*} - s_{4\theta_2^*}\right)\right] + d_5\left[s_{\theta_1^*}c^4_{\theta_1^*} - s_{\theta_2^*}c^4_{\theta_2^*} + \frac{8}{3}\left(s_{\theta_1^*} - s_{\theta_2^*}\right) + \frac{4}{3}\left(s_{\theta_1^*}c^2_{\theta_1^*} - s_{\theta_2^*}c^2_{\theta_2^*}\right)\right]\Bigg\} + \pi K'\left(2e + d_2 + \frac{3}{4}d_4\right) \tag{47}$$

where

$$d_1 = 1 + ec_1 \tag{48}$$

$$d_2 = c_1 + ec_2 \tag{49}$$

$$d_3 = \tfrac{1}{3}(c_2 + ec_3) \tag{50}$$

$$d_4 = c_3 + ec_4 \tag{51}$$

$$d_5 = c_4/5 \tag{52}$$

When shadowing is present, Δe can be either positive or negative, depending on the relative geometry of the orbit and the shadow arc. Finally, the integration of $\mathrm{d}\omega/\mathrm{d}t$ in Eq. (26) can be carried out in closed form without the need for expansions in powers of the eccentricity such that the resulting formula will be valid for any eccentricity $e \geq 10^{-2}$ and not be restricted to the range $10^{-2} \leq e \leq 0.2$, as for Δa and Δe. Therefore

$$\Delta\omega = \frac{2a^2(1 - e^2)^2}{\mu e} f_t \int_0^{2\pi} \frac{s_{\theta^*}}{(1 + ec_{\theta^*})^2\left(1 + e^2 + 2ec_{\theta^*}\right)^{\frac{1}{2}}}\mathrm{d}\theta^* \tag{53}$$

The integral can be written as

$$k'\int \frac{\mathrm{d}x}{(1 + ex)^2(1 + a_1 x)^{\frac{1}{2}}}$$

with a_1 defined as before and with $k' = -1/(1 + e^2)^{1/2}$ and $x = c_{\theta^*}$. However,

$$\int \frac{\mathrm{d}x}{(1 + ex)^2(1 + a_1 x)^{\frac{1}{2}}} = \frac{-1}{k}\left[\frac{(1 + a_1 x)^{\frac{1}{2}}}{(1 + ex)} + \frac{a_1}{2}\int \frac{\mathrm{d}x}{(1 + ex)(1 + a_1 x)^{\frac{1}{2}}}\right]$$

such that after a few additional steps $\Delta\omega$ can be written as

$$\Delta\omega = \frac{-2a^2(1 - e^2)(1 + e^2)^{\frac{1}{2}}}{\mu e^2} f_t \left\{\frac{(1 + a_1 c_{\theta^*})^{\frac{1}{2}}}{(1 + ec_{\theta^*})} + \frac{2}{(1 + e^2)^{\frac{1}{2}}(1 - e^2)^{\frac{1}{2}}}\tan^{-1}\frac{(1 + a_1 c_{\theta^*})^{\frac{1}{2}}}{[(1 - e^2)/(1 + e^2)]^{\frac{1}{2}}}\right\}_0^{2\pi} \tag{54}$$

with $k = -e(1 - e^2)/(1 + e^2)$. These integrations must be carried out from perigee to subsequent perigee. This is needed because $\Delta\omega$ as computed from Eq. (54) modifies the location of the new perigee by a small amount so that, using a common reference direction, small corrections to Δa, Δe, and $\Delta\omega$ must be added to the first-order changes. These corrections are obtained by integrating further between 2π and $\Delta\omega$ by properly accounting for the sign of $\Delta\omega$. From Eq. (54), it follows that, after one full revolution from $\theta^* = 0$ to 2π, $\Delta\omega = 0$, indicating that continuous constant tangential acceleration will not induce any motion in the perigee. This will not be the case if shadowing is present, in which case the integration is carried out from 0 to θ_1^* and from θ_2^* to 2π. The change in ω will be given by

$$\Delta\omega = \frac{2a^2(1 - e^2)}{\mu e^2} f_t \times\left((1 + e^2)\left[\frac{\left(1 + a_1 c_{\theta_2^*}\right)^{\frac{1}{2}}}{1 + ec_{\theta_2^*}} - \frac{\left(1 + a_1 c_{\theta_1^*}\right)^{\frac{1}{2}}}{1 + ec_{\theta_1^*}}\right] + 2(1 - e^2)^{-\frac{1}{2}}\left\{\tan^{-1}\frac{1 + a_1 c_{\theta_2^*}}{[(1 - e^2)/(1 + e^2)]^{\frac{1}{2}}} - \tan^{-1}\frac{1 + a_1 c_{\theta_1^*}}{[(1 - e^2)/(1 + e^2)]^{\frac{1}{2}}}\right\}\right) \tag{55}$$

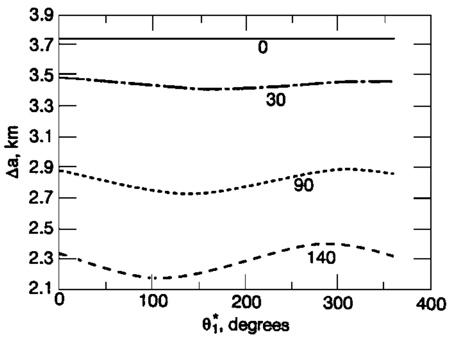

Fig. 1 Semimajor axis increase per revolution vs shadow entry angle for a = 7000 km, e = 0.1, and total shadow angles of 0, 30, 90, and 140 deg.

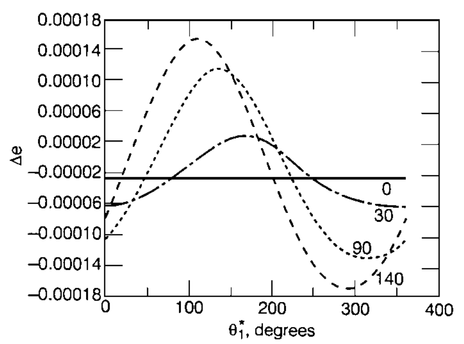

Fig. 2 Eccentricity change per revolution vs shadow entry angle for a = 7000 km, e = 0.1, and total shadow angles of 0, 30, 90, and 140 deg.

Both $1 + a_1 c_{\theta_1^*}$ and $1 + a_1 c_{\theta_2^*}$ are positive quantities, as well as the $[(1 - e^2)/(1 + e^2)]^{1/2}$ term, so that the inverse tangent function will always yield an angle in the first quadrant without ambiguity. It also follows from Eq. (55) that, if $\theta_1^* = \theta_2^*$, then $\Delta\omega = 0$ because in this case the orbit is in its entirety in sunlight. For a given eccentricity and shadow angle, let θ_1^* vary from 0 to 360 deg. The shadow exit angle θ_2^* is computed simply by adding to θ_1^* the shadow angle. Let $a = 7000$ km, $\mu = 398{,}601.3$ km^3/s^2, and $f_t = 3.5 \times 10^{-7}$ km/s^2, which corresponds approximately to an acceleration of $3.5 \times 10^{-5} g$. In Fig. 1, valid for $e = 0.1$, Δa is plotted vs θ_1^* for shadow angles of $\theta_2^* - \theta_1^* = 0$, 30, 90, and 140 deg. For a given shadow angle, the variation is due to the relative geometry of the orbit eccentricity vector and the shadow arc. Because θ^* is measured from perigee, a shadow arc centered at perigee has a smaller length than a comparable shadow arc of equal angular opening centered at apogee. In the latter case, the duration of the thrust is less than for the former so that there will be comparatively less buildup in the semimajor axis. For small eccentricities, this effect is more important than the velocity efficiency effect, which favors thrusting at perigee where velocity is higher, and therefore larger changes in the semimajor axis can be achieved for the same thrust duration than at apogee such as in the impulsive case. This can also be seen from the Δa equation valid for small dt intervals

$$\Delta a = \int \frac{2Va^2}{\mu} f_t \, \mathrm{d}t = \frac{2a^2}{\mu} \int V f_t \, \mathrm{d}t \tag{56}$$

The change in the semimajor axis is independent of the shadow entry angle θ_1^* only if there is no shadow and Δa is at its constant maximum value. As the shadow angle increases, Δa gets smaller with increased amplitude of its oscillation. There is also a shift in the location of the maximum and minimum values with respect to θ_1^* as the shadow angle varies. In Fig. 2, it is shown that $\Delta e > 0$ is maximum if the shadow is centered at apogee, and the largest $\Delta e < 0$ takes place if the shadow is centered at perigee. The term Δe is constant and negative for the case of no shadow as was shown in Eq. (46). The amplitude of the oscillations increases with increasing total shadow angle. From Fig. 3, $\Delta\omega = 0$ if the shadow is centered at either perigee or apogee, as expected due to symmetry; $\Delta\omega = 0$

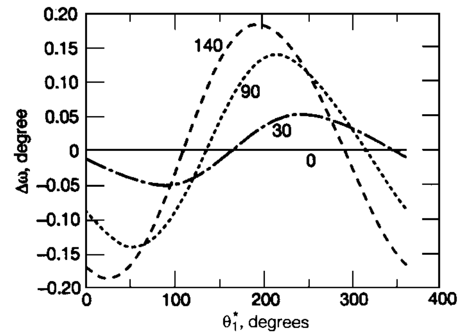

Fig. 3 Argument of perigee change per revolution vs shadow entry angle for a = 7000 km, e = 0.1, and total shadow angles of 0, 30, 90, and 140 deg.

for the no-shadow case with once again increasing amplitude of the oscillations with increasing shadow angle. For $e = 0.01$, the Δa oscillations have smaller overall amplitude variations, whereas the reverse is true for the Δe and $\Delta\omega$ curves, especially for $\Delta\omega$, which reaches maximum changes of the order of 2 deg. For smaller eccentricity, the $\Delta\omega$ equation will break down as the orbit becomes near circular and approaches the $e = 0$ singularity, whereas the Δa and Δe equations are still perfectly valid. For $e < 0.01$ down to $e = 0$, it is preferable and convenient to use the following equations valid for tangential thrusting:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{n} f_t \tag{57}$$

$$\frac{\mathrm{d}e_x}{\mathrm{d}t} = \frac{2a^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} c_{nt} f_t \tag{58}$$

$$\frac{\mathrm{d}e_y}{\mathrm{d}t} = \frac{2a^{\frac{1}{2}}}{\mu^{\frac{1}{2}}} s_{nt} f_t \tag{59}$$

where the x axis is along the line of apsides directed toward perigee, the y axis lies perpendicular to the x axis in the orbit plane, and time is measured from the x axis. Therefore, when e is small, then $n \, \mathrm{d}t = \mathrm{d}\theta$, which is used to convert the preceding equations into the form

$$\frac{\mathrm{d}a}{\mathrm{d}\theta} = \frac{2a^3}{\mu} f_t \tag{60}$$

$$\frac{\mathrm{d}e_x}{\mathrm{d}\theta} = \frac{2a^2}{\mu} f_t c_\theta \tag{61}$$

$$\frac{\mathrm{d}e_y}{\mathrm{d}\theta} = \frac{2a^2}{\mu} f_t s_\theta \tag{62}$$

Integrating between zero and θ_1 and between θ_2 and 2π and adding results in

$$\Delta a = (2a^3/\mu) f_t (2\pi + \theta_1 - \theta_2) \tag{63}$$

$$\Delta e_x = (2a^2/\mu) f_t \left(s_{\theta_1} - s_{\theta_2}\right) \tag{64}$$

$$\Delta e_y = (-2a^2/\mu) f_t \left(c_{\theta_1} - c_{\theta_2}\right) \tag{65}$$

Because perigee is initially along the x axis, the change in ω or $\Delta\omega$ is simply given by

$$\Delta\omega = \tan^{-1}\left(\frac{\Delta e_y}{e + \Delta e_x}\right) \tag{66}$$

which is always well defined. The change in e or Δe can also be given in this near-circular case by

$$\Delta e = \left[(e + \Delta e_x)^2 + \Delta e_y^2\right]^{\frac{1}{2}} - e \tag{67}$$

The secular effect of J_2 on the elements Ω, ω, and M can be added analytically after each revolution regardless of the existence of a shadow arc.

For the problem of multirevolution orbit prediction in the absence of Earth shadow using continuous constant acceleration tangential thrusting, analytic expressions for the orbit elements as a function of time valid for several orbits at the geostationary Earth orbit (GEO) level and several tens of orbits at the LEO level can be developed. This theory is compared next with an exact numerically integrated trajectory involving only the tangential thrust. Because the eccentricity buildup with continuous tangential thrusting is rather small, the analytic integration of the orbital equations linearized about zero eccentricity provides solutions that do not break down rapidly provided that the mean motion is not held constant during the integration. This analytic integration is further extended by the inclusion of the J_2 effect during the multirevolution orbit expansion.

Analytic Integration for Near-Circular Orbits with Continuous Thrust

The full set of the Gaussian form of the Lagrange planetary equations for near-circular orbits is given by

$$\frac{\mathrm{d}a}{\mathrm{d}t} = 2a\left(\frac{f_t}{V}\right) \tag{68}$$

$$\frac{\mathrm{d}e_x}{\mathrm{d}t} = \left(\frac{2f_t}{V}\right)\cos\alpha - \left(\frac{f_n}{V}\right)\sin\alpha \tag{69}$$

$$\frac{\mathrm{d}e_y}{\mathrm{d}t} = 2\left(\frac{f_t}{V}\right)\sin\alpha + \left(\frac{f_n}{V}\right)\cos\alpha \tag{70}$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \left(\frac{f_h}{V}\right)\cos\alpha \tag{71}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \left(\frac{f_h}{V}\right)\frac{\sin\alpha}{\sin i} \tag{72}$$

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = n + \frac{2f_n}{V} - \left(\frac{f_h}{V}\right)\frac{\sin\alpha}{\tan i} \tag{73}$$

The classical orbit elements being a, e, i, Ω, ω, and M, the preceding elements make use of the nonsingular parameters $e_x = e\cos\omega$, $e_y = e\sin\omega$, and $\alpha = \omega + M$, which represents the mean longitude. Furthermore, $n = (\mu/a^3)^{1/2}$ represents the orbit mean motion, $V = na = (\mu/a)^{1/2}$ the circular velocity, and f_t, f_n, and f_h the components of the thrust acceleration vector along the tangent, normal, and out-of-plane directions, respectively, with the normal direction now oriented toward the center of attraction and lying in the instantaneous orbit plane. If we assume only tangential thrusting, then the preceding set of equations reduces to

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2}{n}f_t \tag{74}$$

$$\frac{\mathrm{d}e_x}{\mathrm{d}t} = \frac{2}{na}f_t\cos\alpha \tag{75}$$

$$\frac{\mathrm{d}e_y}{\mathrm{d}t} = \frac{2}{na}f_t\sin\alpha \tag{76}$$

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = n \tag{77}$$

Equation (74) for the semimajor axis can be integrated in a straightforward manner for f_t constant:

$$\int_{a_0}^{a} a^{-\frac{3}{2}}\,\mathrm{d}a = \frac{2}{\mu^{\frac{1}{2}}}f_t\int_0^t \mathrm{d}t, \qquad a^{-\frac{1}{2}} = a_0^{-\frac{1}{2}} - \frac{f_t}{\mu^{\frac{1}{2}}}t \tag{78}$$

which can also be cast into the form

$$a = \frac{1}{At^2 + Bt + C} \tag{79}$$

where

$$A = \left(f_t/\mu^{\frac{1}{2}}\right)^2 \tag{80}$$

$$B = -\frac{2f_t}{(a_0\mu)^{\frac{1}{2}}} \tag{81}$$

$$C = 1/a_0 \tag{82}$$

At time $t = 0$, $a = a_0$, and as a approaches infinity, t will tend to

$$t = \frac{\mu^{\frac{1}{2}}}{f_t a_0^{\frac{1}{2}}} \tag{83}$$

as can be seen by setting Eq. (78) to zero. This result is valid as long as the orbit remains near circular. Equation (77) for the mean longitude can now be integrated because

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = n = \mu^{\frac{1}{2}}a^{-\frac{3}{2}} \tag{84}$$

where from Eq. (78)

$$a^{-\frac{3}{2}} = \left[a_0^{-\frac{1}{2}} - \left(f_t/\mu^{\frac{1}{2}}\right)t\right]^3$$

and therefore

$$\int_{\alpha_0}^{\boldsymbol{\alpha}} \mathrm{d}\alpha = \int_0^t \mu^{\frac{1}{2}}\left[a_0^{-\frac{3}{2}} - \frac{3}{a_0}\left(\frac{f_t}{\mu^{\frac{1}{2}}}\right)t + 3a_0^{-\frac{1}{2}}\left(\frac{f_t}{\mu^{\frac{1}{2}}}\right)^2 t^2 - \left(\frac{f_t}{\mu^{\frac{1}{2}}}\right)^3 t^3\right]\mathrm{d}t$$

resulting in

$$\alpha = \alpha_0 + n_0 t + b't^2 + c't^3 + d't^4 \tag{85}$$

with

$$b' = -(3/2a_0)f_t \tag{86}$$

$$c' = \frac{a_0^{-\frac{1}{2}}f_t^2}{\mu^{\frac{1}{2}}} \tag{87}$$

$$d' = -f_t^3/4\mu \tag{88}$$

An expression for α as a function of the semimajor axis can also be obtained by dividing Eq. (77) by Eq. (74):

$$\frac{\mathrm{d}\alpha}{\mathrm{d}a} = \frac{\mu a^{-3}}{2f_t}$$

which yields upon integration

$$\alpha = \alpha_0 - (\mu/4f_t)\left(a^{-2} - a_0^{-2}\right) \tag{89}$$

This expression can readily be solved for a in terms of α:

$$a = \left[\frac{1}{a_0^{-2} - (4f_t/\mu)(\alpha - \alpha_0)}\right]^{\frac{1}{2}} \tag{90}$$

The integration of Eq. (75) is best carried out by dividing Eq. (75) by Eq. (77) such that $\mathrm{d}e_x/\mathrm{d}\alpha = (2f_t a^2/\mu)c_{\boldsymbol{\alpha}}$, and using a from Eq. (90), the preceding derivative can be written as

$$\mathrm{d}e_x = [(2f_t/\mu)/(b + g\alpha)]c_{\boldsymbol{\alpha}}\,\mathrm{d}\alpha$$

with

$$g = -4f_t/\mu \tag{91}$$

$$b = a_0^{-2} + (4f_t/\mu)\alpha_0 \tag{92}$$

such that

$$\mathrm{d}e_x = -\tfrac{1}{2}(g/b)[1 + (g/b)\alpha]^{-1} c_{\boldsymbol{\alpha}}\,\mathrm{d}\alpha \tag{93}$$

The quantity g/b is on the order of 10^{-4} at LEO and 10^{-3} at GEO for a thrust acceleration of the order $10^{-5}\,g$. Therefore, we can expand the term $(1 + [g/b]\alpha)^{-1}$ before integrating. This results in

$$\mathrm{d}e_x = -\tfrac{1}{2}(\delta - \delta^2\alpha + \delta^3\alpha^2 - \delta^4\alpha^3 + \delta^5\alpha^4 - \delta^6\alpha^5 + \cdots)c_{\boldsymbol{\alpha}}\,\mathrm{d}\alpha$$

where δ stands for the small quantity

$$\delta = g/b \tag{94}$$

Finally, and after integration,

$$\begin{aligned} e_x = e_{x_0} - \tfrac{1}{2}\Big\{&\delta\big(s_{\boldsymbol{\alpha}} - s_{\boldsymbol{\alpha}_0}\big) - \delta^2\Big[\big(c_{\boldsymbol{\alpha}} - c_{\boldsymbol{\alpha}_0}\big) + \big(\alpha s_{\boldsymbol{\alpha}} - \alpha_0 s_{\boldsymbol{\alpha}_0}\big)\Big] \\ &+ \delta^3\Big[2\alpha c_{\boldsymbol{\alpha}} - 2\alpha_0 c_{\boldsymbol{\alpha}_0} + (\alpha^2 - 2)s_{\boldsymbol{\alpha}} - \big(\alpha_0^2 - 2\big)s_{\boldsymbol{\alpha}_0}\Big] \\ &- \delta^4\Big[(3\alpha^2 - 6)c_{\boldsymbol{\alpha}} - \big(3\alpha_0^2 - 6\big)c_{\boldsymbol{\alpha}_0} + (\alpha^3 - 6\alpha)s_{\boldsymbol{\alpha}} \\ &- \big(\alpha_0^3 - 6\alpha_0\big)s_{\boldsymbol{\alpha}_0}\Big]\Big\} \end{aligned} \tag{95}$$

In a similar way, dividing Eq. (76) by Eq. (77) results in

$$\begin{aligned} \mathrm{d}e_y &= -\tfrac{1}{2}(g/b)[1 + (g/b)\alpha]^{-1} s_{\boldsymbol{\alpha}}\,\mathrm{d}\alpha \\ &= -\tfrac{1}{2}(\delta - \delta^2\alpha + \delta^3\alpha^2 - \delta^4\alpha^3 + \delta^5\alpha^4 - \delta^6\alpha^5 + \cdots)s_{\boldsymbol{\alpha}}\,\mathrm{d}\alpha \end{aligned}$$

which upon integration yields

$$\begin{aligned} e_y = e_{y_0} - \tfrac{1}{2}\Big\{&-\delta\big(c_{\boldsymbol{\alpha}} - c_{\boldsymbol{\alpha}_0}\big) - \delta^2\Big[\big(s_{\boldsymbol{\alpha}} - s_{\boldsymbol{\alpha}_0}\big) - \big(\alpha c_{\boldsymbol{\alpha}} - \alpha_0 c_{\boldsymbol{\alpha}_0}\big)\Big] \\ &+ \delta^3\Big[2\alpha s_{\boldsymbol{\alpha}} - 2\alpha_0 s_{\boldsymbol{\alpha}_0} - (\alpha^2 - 2)c_{\boldsymbol{\alpha}} + \big(\alpha_0^2 - 2\big)c_{\boldsymbol{\alpha}_0}\Big] \\ &- \delta^4\Big[(3\alpha^2 - 6)s_{\boldsymbol{\alpha}} - \big(3\alpha_0^2 - 6\big)s_{\boldsymbol{\alpha}_0} - (\alpha^3 - 6\alpha)c_{\boldsymbol{\alpha}} \\ &+ (\alpha_0^3 - 6\alpha_0)c_{\boldsymbol{\alpha}_0}\Big]\Big\} \end{aligned} \tag{96}$$

The analytic solution of the system of Eqs. (74–77) is given by Eq. (78) for the semimajor axis, Eq. (85) for the mean longitude α, and Eqs. (95) and (96) for the eccentricity vector components e_x and e_y. The semimajor axis and mean longitude are integrated exactly and are obtained explicitly as functions of time. However, e_x and e_y are given as a function of α and not time and are approximate because they involve expansions. These expressions are valid for many revolutions as will be shown later and are more accurate than the linearized equations developed earlier because the mean motion n and the semimajor axis are varying in Eqs. (74–77). The linearized equations hold a and n constant in the right-hand side of these equations, which are then integrated readily to yield

$$a = a_0 + (2/n)f_t t \tag{97}$$

$$e_x = e_{x_0} + \big(2a^{\frac{1}{2}}\big/\mu^{\frac{1}{2}}\big)(f_t/n)s_{nt} \tag{98}$$

$$e_y = e_{y_0} + \big(2a^{\frac{1}{2}}\big/\mu^{\frac{1}{2}}\big)(f_t/n)(1 - c_{nt}) \tag{99}$$

Of course, $\alpha = nt$ was used to integrate the equations of motion, whereas in Eq. (85) α is not linear but quartic in time, which is the exact functional dependence for large t. That is why Eqs. (78), (85), (95), and (96) will not break down as fast as the linearized Eqs. (97–99), which would be accurate for very few revolutions.

A more accurate description of the motion requires the consideration of the J_2 effect on the various elements, in particular e_x, e_y, Ω, and α because a and i do not exhibit any secular variations due to J_2. Using the near-circular assumption, the J_2 accelerations can be written as

$$(f_n)_{J_2} = \frac{3\mu J_2 R^2}{2a^4}\big(1 - 3s_i^2 s_\theta^2\big) \tag{100}$$

$$(f_t)_{J_2} = \frac{-3\mu J_2 R^2}{a^4} s_i^2 s_\theta c_\theta \tag{101}$$

$$(f_h)_{J_2} = \frac{-3\mu J_2 R^2}{a^4} s_i c_i s_\theta \tag{102}$$

and with $V = \mu^{1/2}a^{-1/2}$ and $\alpha = \theta = nt$ with time measured from the ascending node. The regression of the node Ω is obtained from the variational equation given in Eq. (72):

$$\Omega = \Omega_0 - 3\mu^{\frac{1}{2}} J_2 R^2 c_i \int_0^t \left(a_0^{-\frac{1}{2}} - \frac{f_t}{\mu^{\frac{1}{2}}}t\right)^7 s_{nt}^2\,\mathrm{d}t$$

From

$$n = \mu^{\frac{1}{2}} a^{-\frac{3}{2}} = \mu^{\frac{1}{2}}\Big[a_0^{-\frac{1}{2}} - \big(f_t\big/\mu^{\frac{1}{2}}\big)t\Big]^3$$

we have $nt = n_0(1 - \varepsilon t)^3 t$, where $n_0 = \mu^{1/2}a_0^{-3/2}$ and $\varepsilon = f_t/(\mu^{1/2}a_0^{-1/2})$. For our LEO example at hand, namely, $a_0 = 7000$ km, and $f_t = 3.5 \times 10^{-7}$ km/s^2, ε is on the order of 10^{-8} such that $nt \cong n_0 t(1 - 3\varepsilon t)$. Keeping only first-order terms in ε, $s_{nt} \cong s_{n_0 t} - 3\varepsilon n_0 t^2 c_{n_0 t}$ and $s_{nt}^2 \cong s_{n_0 t}^2 - 6\varepsilon n_0 t^2 s_{n_0 t} c_{n_0 t}$ are obtained after making use of the expansion $\sin(x + \varepsilon') \cong s_x + \varepsilon' c_x$ valid for small ε'. The preceding integration can be carried out as

$$\begin{aligned} \Omega &= \Omega_0 - 3\mu^{\frac{1}{2}} J_2 R^2 c_i a_0^{-\frac{7}{2}} \int_0^t (1 - 7\varepsilon t)s_{nt}^2\,\mathrm{d}t \\ &= \Omega_0 - \frac{3}{2}\mu^{\frac{1}{2}} J_2 R^2 c_i a_0^{-\frac{7}{2}}\Bigg[t - \frac{\varepsilon}{2n_0^2}\left(\frac{1}{2} + 7n_0^2 t^2\right) \\ &\quad - \frac{1}{2n_0}(1 - \varepsilon t)s_{2n_0 t} + \frac{\varepsilon}{2n_0^2}\left(\frac{1}{2} + 6n_0^2 t^2\right)c_{2n_0 t}\Bigg] \end{aligned} \tag{103}$$

In the absence of any thrust, $\varepsilon = 0$ and the orbit will not expand such that the secular variation $\Omega = \Omega_0 - \frac{3}{2}(n_0/a_0^2)J_2 R^2 c_i t$ is recovered from Eq. (103). In a similar way, Eq. (73) for $\dot\alpha$ can be integrated using f_n and f_h given in Eqs. (100) and (102), respectively. Because $b' = \frac{3}{2}n_0\varepsilon$ is first order in ε, whereas $c' = n_0\varepsilon^2$ and d' in Eq. (85) are of higher order, the analytic integration of $\dot\alpha$ retaining only first-order terms in ε yields

$$\begin{aligned} \alpha &= \big(\alpha_0 + n_0 t + b' t^2\big) + 3\mu^{\frac{1}{2}} J_2 R^2 a_0^{-\frac{7}{2}}\big(t - \tfrac{7}{2}\varepsilon t^2\big) \\ &\quad + 3\mu^{\frac{1}{2}} J_2 R^2 a_0^{-\frac{7}{2}}\left(\frac{1 - 4s_i^2}{2}\right)\Bigg[t - \frac{\varepsilon}{2n_0^2}\left(\frac{1}{2} + 7n_0^2 t^2\right) \\ &\quad - \frac{1}{2n_0}(1 - \varepsilon t)s_{2n_0 t} + \frac{\varepsilon}{2n_0^2}\left(\frac{1}{2} + 6n_0^2 t^2\right)c_{2n_0 t}\Bigg] \end{aligned} \tag{104}$$

with the terms in the first parentheses due to the thrust only. In the absence of any thrust, i.e., $\varepsilon = 0$, Eq. (104) yields $\dot\alpha = \dot\omega + \dot M = n_0 + (3\mu^{1/2}/2)J_2 R^2 a_0^{-7/2}(3 - 4s_i^2)$, the combined secular variations of ω and M, namely, ω_s and M_s given by $\omega_s = \omega_0 + \frac{3}{2}J_2 R^2 \bar n a_0^{-2}(2 - \frac{5}{2}s_i^2)t$ and $M_s = M_0 + \bar n t$, respectively, where $\bar n = n_0 + \frac{3}{2}(J_2 R^2/a_0^2)(1 - \frac{3}{2}s_i^2)n_0$.

Finally, the J_2 contribution to e_x and e_y is evaluated by direct integration of the rates given in Eqs. (69) and (70). Thus

$$\frac{\mathrm{d}e_x}{\mathrm{d}t} = -2\big(3\mu^{\frac{1}{2}} J_2 R^2 a^{-\frac{7}{2}}\big)s_i^2 s_\theta c_\theta^2 - \frac{3\mu^{\frac{1}{2}}}{2} J_2 R^2 a^{-\frac{7}{2}}\big(1 - 3s_i^2 s_\theta^3\big)$$

and using

$$s_\theta^3 \cong s_{nt}^3 \approx s_{n_0t}^3 - 9\varepsilon n_0 t^2 s_{n_0t}^2 c_{n_0t}$$

and now letting $C = \mu^{1/2} J_2 R^2 a_0^{-7/2}$, we have

$$\frac{\mathrm{d}e_x}{\mathrm{d}t} = -\left(\frac{3}{2} + 6s_i^2\right)C(1-7\varepsilon t)s_{nt} + \frac{21}{2}Cs_i^2(1-7\varepsilon t)s_{nt}^3$$

$$e_x = e_{x_0} - \left(\frac{3}{2} + 6s_i^2\right)C\left[\frac{1}{n_0}\left(1 - c_{n_0t}\right) - \frac{\varepsilon}{n_0^2}s_{n_0t} + \frac{\varepsilon t}{n_0}c_{n_0t} - 3\varepsilon t^2 s_{n_0t}\right] + \frac{21}{2}Cs_i^2\left[\frac{2}{3n_0}\left(1 - c_{n_0t}\right) - \frac{1}{3n_0}c_{n_0t}s_{n_0t}^2 - \frac{9}{4}\varepsilon t^2 s_{n_0t} + \frac{3}{4}\varepsilon t^2 s_{3n_0t} - \frac{\varepsilon}{12n_0}tc_{3n_0t} + \frac{\varepsilon}{36n_0^2}s_{3n_0t} + \frac{3}{4}\frac{\varepsilon}{n_0}tc_{n_0t} - \frac{3}{4}\frac{\varepsilon}{n_0^2}s_{n_0t}\right] \tag{105}$$

The thrust-only contribution can also be obtained as a function of time such that, with $a^{-1/2} = a_0^{-1/2} - (f_t/\mu^{1/2})t$,

$$(\dot{e}_x)_T = \frac{2f_t}{\mu^{\frac{1}{2}}a^{\frac{1}{2}}}c_{\boldsymbol{\alpha}} \cong \frac{2f_t}{\mu^{\frac{1}{2}}}a_0^{-\frac{1}{2}}(1-\varepsilon t)c_{nt}$$

with $c_{nt} = \cos(n_0t - 3\varepsilon n_0 t^2) \cong c_{n_0t} + 3\varepsilon n_0 t^2 s_{n_0t}$ and $C' = (2f_t/\mu^{1/2})a_0^{-1/2}$,

$$(e_x)_T = C'\left[(1/n_0)s_{n_0t} + (5\varepsilon/n_0)ts_{n_0t} - 3\varepsilon t^2 c_{n_0t} + \left(5\varepsilon/n_0^2\right)c_{n_0t} - \left(5\varepsilon/n_0^2\right)\right] \tag{106}$$

From $(\dot{e}_y)_T = (2f_t/\mu^{1/2}a^{1/2})s_{nt}$, the thrust-only contribution to e_y is also obtained as

$$(e_y)_T = C'\left[(1/n_0)(1 - c_{n_0t}) - (5\varepsilon/n_0)tc_{n_0t} - 3\varepsilon t^2 s_{n_0t} + \left(5\varepsilon/n_0^2\right)s_{n_0t}\right] \tag{107}$$

Finally, the J_2 effect on the element e_y is obtained from Eq. (70) such as

$$\dot{e}_y = -\tfrac{21}{2}Cs_i^2(1-7\varepsilon t)c_{nt}s_{nt}^2 + \tfrac{3}{2}C(1-7\varepsilon t)c_{nt}$$

with $c_{nt}s_{nt}^2 \cong c_{n_0t}s_{n_0t}^2 - 6\varepsilon n_0 t^2 s_{n_0t}c_{n_0t}^2 + 3\varepsilon n_0 t^2 s_{n_0t}^3$, yielding

$$e_y = e_{y_0} - \frac{21}{2}Cs_i^2\left(\frac{1}{3n_0}s_{n_0t} - \frac{1}{3n_0}s_{n_0t}c_{n_0t}^2 - \frac{\varepsilon}{4n_0}ts_{n_0t} - \frac{3\varepsilon}{4}t^2 c_{n_0t} - \frac{\varepsilon}{4n_0^2}c_{n_0t} + \frac{2\varepsilon}{9n_0^2} + \frac{3}{4}\varepsilon t^2 c_{3n_0t} + \frac{\varepsilon t}{12n_0}s_{3n_0t} + \frac{\varepsilon}{36n_0^2}c_{3n_0t}\right) + \frac{3}{2}C\left(\frac{1}{n_0}s_{n_0t} - 3\varepsilon t^2 c_{n_0t} - \frac{\varepsilon}{n_0^2}c_{n_0t} - \frac{\varepsilon}{n_0}ts_{n_0t} + \frac{\varepsilon}{n_0^2}\right) \tag{108}$$

Thus, the combined effects of J_2 and the thrust acceleration yield

$$(e_x)_{\text{tot}} = (e_x)_T + e_x \tag{109}$$

$$(e_y)_{\text{tot}} = (e_y)_T + e_y \tag{110}$$

where $(e_x)_T$, e_x, $(e_y)_T$, and e_y are given by Eqs. (106), (105), (107), and (108), respectively. These equations remain accurate as long as the orbit stays near circular. In this case the effect of the drag acceleration is accounted for by using an effective thrust acceleration because the drag perturbation will act opposite to the thrust acceleration.

Comparison with a Numerically Integrated Exact Nonsingular Set for Continuous Thrust

The equations of motion for the thrust perturbation case, Eqs. (68–73), are essentially linearized about a circular orbit. It is for this reason that e_x and e_y will lose accuracy over the long time period. To assess the accuracy of the analytic expressions developed earlier, we need to know the exact behavior of the trajectory, and this can only be obtained by numerically integrating the set of exact first-order differential equations using the nonsingular elements e_x and e_y instead of the classical elements e and ω. These equations are given by

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2Vf_t}{an^2} \tag{111}$$

$$\frac{\mathrm{d}e_x}{\mathrm{d}t} = \frac{2f_t(e_x + \cos\alpha_{\theta^*}) - f_n[(r/a)\sin\alpha_{\theta^*} + 2e_y]}{V} + \frac{e_y r\sin\alpha_{\theta^*} f_h}{na^2(1-e^2)^{\frac{1}{2}}\tan i} \tag{112}$$

$$\frac{\mathrm{d}e_y}{\mathrm{d}t} = \frac{2f_t(e_y + \sin\alpha_{\theta^*}) + f_n[(r/a)\cos\alpha_{\theta^*} + 2e_x]}{V} - \frac{e_x r\sin\alpha_{\theta^*} f_h}{na^2(1-e^2)^{\frac{1}{2}}\tan i} \tag{113}$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{r\cos\alpha_{\theta^*} f_h}{na^2(1-e^2)^{\frac{1}{2}}} \tag{114}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{r\sin\alpha_{\theta^*} f_h}{na^2(1-e^2)^{\frac{1}{2}}\sin i} \tag{115}$$

$$\frac{\mathrm{d}(\omega + M)}{\mathrm{d}t} = n + \left(\frac{2f_t\sin\theta^*}{Ve}\right) \times \left\{1 - (1-e^2)^{\frac{1}{2}}\left[1 + e^2(1 + ec_{\theta^*})^{-1}\right]\right\} + \frac{f_n r\left[2e + \cos\theta^* + e^2\cos\theta^* - (1-e^2)^{\frac{3}{2}}\cos\theta^*\right]}{Vae(1-e^2)} - \frac{r\sin\alpha_{\theta^*} f_h}{na^2\tan i(1-e^2)^{\frac{1}{2}}} \tag{116}$$

Here θ^* is the true anomaly, $r = a(1-e^2)(1+ec_{\theta^*})^{-1}$ the radial distance to the spacecraft, $V = \{\mu(1 + e^2 + 2ec_{\theta^*})/[a(1-e^2)]\}^{1/2}$ the orbital velocity on the elliptic orbit, and $\alpha_{\theta^*} = \omega + \theta^* = \alpha + \theta^* - M$ the mean longitude of the spacecraft measured from the ascending node, with $\alpha = \omega + M$ the mean longitude as before. For coplanar transfers, $f_n = f_h = 0$ such that the full set of the coupled nonlinear differential equations reduces to

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2Vf_t}{n^2 a} \tag{117}$$

$$\frac{\mathrm{d}e_x}{\mathrm{d}t} = \frac{2f_t(e_x + \cos\alpha_{\theta^*})}{V} \tag{118}$$

$$\frac{\mathrm{d}e_y}{\mathrm{d}t} = \frac{2f_t(e_y + \sin\alpha_{\theta^*})}{V} \tag{119}$$

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = n + 2f_t\sin\theta^*\left[1 - (1-e^2)^{\frac{1}{2}}\left(1 + \frac{e^2}{1 + e\cos\theta^*}\right)\right]\Big/ Ve \tag{120}$$

Given a, e_x, e_y, and α at time zero, the related quantities needed in the evaluation of the preceding derivatives are computed from $\omega = \tan^{-1}(e_y/e_x)$, $M = \alpha - \omega$. The eccentric anomaly is calculated from Kepler's equation $M = E - e\sin E$; the true anomaly obtained

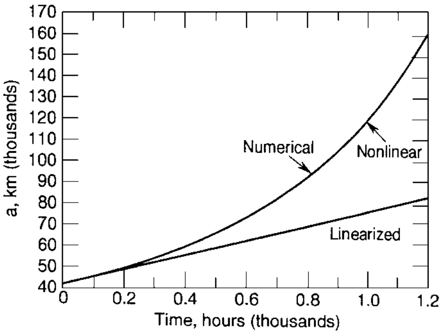

Fig. 4 Evolution of semimajor axis in time for continuous tangential thrust with initial a_0 = 42,000 km and e_0 = 0, using numerical, analytic, and linearized methods.

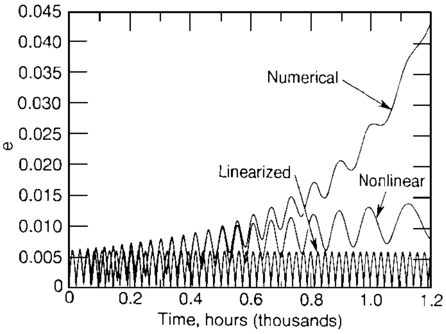

Fig. 5 Evolution of eccentricity magnitude in time for continuous tangential thrust with initial a_0 = 42,000 km and e_0 = 0, using numerical, analytic, and linearized methods.

from $\tan(\theta^*/2) = [(1+e)/(1-e)]^{1/2}\tan(E/2)$; and finally $\alpha_{\theta^*} = \alpha + \theta^* - M$. The differential Eqs. (117–120) are integrated forward in time starting from GEO at $a_0 = 42{,}000$ km, $e_{x_0} = e_{y_0} = 0$, $\alpha_0 = 0$ with $f_t = 3.5 \times 10^{-7}$ km/s^2. A very long time span is considered in Figs. 4 and 5 corresponding to 50 revolutions or some 1200 h. The analytic expressions in Eqs. (78), (85), (95), and (96) for the nonlinear theory, as well as Eqs. (97–99) of the linearized theory, are plotted against the exact numerical solution of Eqs. (117–120). In Fig. 4, the nonlinear and numerical curves are identical as expected because $\mathrm{d}a/\mathrm{d}t$ is insensitive to the eccentricity in both formulations. The evolution of e_x shows that the nonlinear analytic theory matches the exact trajectory for about 10 revolutions in this high-energy orbit and stays in phase with it, whereas the linearized theory breaks down after 1 or 2 revolutions in both phase and amplitude.

These observations are valid for the e_y component as well. Figure 5 shows the evolution of the eccentricity, displaying excellent agreement between the analytic nonlinear and the numerical solution up to $e = 10^{-2}$, after which the orbit is no longer near circular so that only the integrated orbit is meaningful. However, for lower orbits, the agreement will last much longer or on the order of several tens of revolutions. Orbit prediction can therefore be accomplished over long time periods by way of analytic expressions developed here as long as the orbit remains near circular, after which analytic expressions such as those developed earlier and based on the classical elements formulation can be used for larger elliptic orbits by updating the elements one orbit at a time.

Analytic Integration in Near-Circular Orbit with Respect to the Mean Motion: A Series Solution

In this section, we revisit Eqs. (74–77), and instead of solving them as a function of the mean longitude α, we integrate instead with respect to the mean motion $n = \mu^{1/2}a^{-3/2}$. Equation (74) is integrated as before to yield Eq. (78), which reads as $a^{-1/2} = a_0^{-1/2} - (f_t/\mu^{1/2})t$. Multiplying both sides by $a_0^{-1}\mu^{1/2}/n_0$ yields

$$k_4 n^{\frac{1}{3}} = 1 - k_3 t \tag{121}$$

where k_3 and k_4 are constants given by $k_4 = (\mu^{1/3}/a_0 n_0)$ and $k_3 = (f_t/a_0 n_0)$. Equation (121) yields $\frac{1}{3}k_4 n^{-2/3}\,\mathrm{d}n = -k_3\,\mathrm{d}t$, which can be used to eliminate time in favor of n, the mean motion. Using this change of variable, Eq. (77), namely, $\mathrm{d}\alpha/\mathrm{d}t = n$, yields

$$\frac{\mathrm{d}\alpha}{\mathrm{d}n} = -n^{\frac{1}{3}}\frac{k_4}{3k_3} = k_5 n^{\frac{1}{3}} \tag{122}$$

where $k_5 = -k_4/3k_3$. Equation (122) is now readily integrated as

$$\int_{\alpha_0}^{\alpha}\mathrm{d}\alpha = k_5\int_{n_0}^{n} n^{\frac{1}{3}}\mathrm{d}n$$

resulting in an expression for the mean longitude α in terms of the mean motion n:

$$\alpha = \alpha_0 + k_6\left(n^{\frac{4}{3}} - n_0^{\frac{4}{3}}\right) \tag{123}$$

where $k_6 = 3k_5/4$. Now from $n = \mu^{1/2}a^{-3/2}$, we have $\mathrm{d}n = -\frac{3}{2}\mu^{1/2}a^{-5/2}\,\mathrm{d}a$, which allows us to write Eqs. (75) and (76) in terms of n by first dividing by Eq. (74):

$$\frac{\mathrm{d}e_x}{\mathrm{d}t}\bigg/\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{\mathrm{d}e_x}{\mathrm{d}a} = \frac{\cos\alpha}{a}, \qquad \frac{\mathrm{d}e_y}{\mathrm{d}t}\bigg/\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{\mathrm{d}e_y}{\mathrm{d}a} = \frac{\sin\alpha}{a}$$

and therefore

$$\frac{\mathrm{d}e_x}{\mathrm{d}n} = -\frac{2}{3}\frac{\cos\alpha}{n}, \qquad \frac{\mathrm{d}e_y}{\mathrm{d}n} = -\frac{2}{3}\frac{\sin\alpha}{n}$$

Because α is itself known from Eq. (123) as a function of n, the preceding two derivatives can be cast into the following form:

$$\frac{\mathrm{d}e_x}{\mathrm{d}n} = -\frac{2}{3n}\cos\left[\alpha_0 + k_6\left(n^{\frac{4}{3}} - n_0^{\frac{4}{3}}\right)\right]$$

$$\frac{\mathrm{d}e_y}{\mathrm{d}n} = -\frac{2}{3n}\sin\left[\alpha_0 + k_6\left(n^{\frac{4}{3}} - n_0^{\frac{4}{3}}\right)\right]$$

Let us now carry out a change of variable from n to m with $n^{4/3} = m$ and $\mathrm{d}m = \frac{4}{3}n^{1/3}\,\mathrm{d}n$. This will yield, after some manipulations,

$$\mathrm{d}e_x = -\frac{1}{2}\left(A\frac{\cos k_6 m}{m} - B\frac{\sin k_6 m}{m}\right)\mathrm{d}m \tag{124}$$

$$\mathrm{d}e_y = -\frac{1}{2}\left(B\frac{\cos k_6 m}{m} + A\frac{\sin k_6 m}{m}\right)\mathrm{d}m \tag{125}$$

with $A = \cos(\alpha_0 - k_6 n_0^{4/3})$ and $B = \sin(\alpha_0 - k_6 n_0^{4/3})$. Integrating Eqs. (124) and (125) yields

$$e_x = e_{x_0} - \frac{A}{2}\left\{\ln\left(\frac{m}{m_0}\right) + \sum_{\ell=1}^{\infty}\frac{(-1)^\ell (k_6)^{2\ell}}{2\ell(2\ell)!}\left[(m)^{2\ell} - (m_0)^{2\ell}\right]\right\}$$

$$+\frac{B}{2}\left\{\sum_{\ell=0}^{\infty}\frac{(-1)^\ell (k_6)^{2\ell+1}}{(2\ell+1)(2\ell+1)!}\left[(m)^{2\ell+1} - (m_0)^{2\ell+1}\right]\right\} \tag{126}$$

$$e_y = e_{y_0} - \frac{B}{2}\left\{\ln\left(\frac{m}{m_0}\right) + \sum_{\ell=1}^{\infty}\frac{(-1)^\ell (k_6)^{2\ell}}{2\ell(2\ell)!}\left[(m)^{2\ell} - (m_0)^{2\ell}\right]\right\}$$

$$-\frac{A}{2}\left\{\sum_{\ell=0}^{\infty}\frac{(-1)^\ell (k_6)^{2\ell+1}}{(2\ell+1)(2\ell+1)!}\left[(m)^{2\ell+1} - (m_0)^{2\ell+1}\right]\right\} \tag{127}$$

Because the semimajor axis is known as a function of time t from Eq. (78), the mean longitude α is obtained from Eq. (123) as a function of the mean motion, and therefore the semimajor axis and finally both e_x and e_y are evaluated from Eqs. (126) and (127) in terms of the mean motion too after truncating the series at an appropriate value of ℓ. These equations will yield the same medium- to long-term accuracy as the theory depicted by Eqs. (78), (85), (95), and (96) of the preceding section.

Concluding Remarks

An orbit prediction capability based on analytic expressions that describe the evolution of the in-plane orbit elements with high accuracy is presented for the problem of low-thrust tangential thrusting along small-to-moderate eccentricity orbits. The orbit is updated after each revolution if shadow is present to account for the changing shadow geometry, and the secular variation of the node due to J_2 is updated analytically. Analytic expressions for the variations of the pertinent elements due to the combined effects of J_2 and the thrust acceleration are also obtained for the near-circular case in LEO using continuous, constant, low-thrust acceleration along the tangential direction. The analytic modelings of the coplanar phase of a typical LEO-to-GEO transfer allow for straightforward implementation in fast simulation computer programs to support systems design optimization analysis.

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